HOSSAM GHANEM

(15) 7.7 Indeterminate Forms and I 'Hopital's Rule (A)

Using Logarithmic To Find Limits:

$\ln \lim_{x\to a} f(x) = \lim_{x\to a} \ln f(x)$		*IF* $\ln \lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a} f(x) = e^L$	
$e^{\ln a} = a$	$e^0 = 1$	$e^{\infty} = \infty$	$e^{-\infty} = 0$

Indeterminate form:

Indeterminate form	$\frac{0}{0}$	$\frac{\pm\infty}{\pm\infty}$	$0\cdot\pm\infty$	0^0	∞^0	1^{∞}	$\infty-\infty$

$\frac{0}{0}$ $\frac{\pm\infty}{\pm\infty}$ Guidelines	$L = \lim_{x\to a} \frac{f(x)}{g(x)} \to \frac{0}{0}$	$L = \lim_{x\to a} \frac{f'(x)}{g'(x)}$	يمكن التكرار في حالة استمرار النتيجة
Example	$L = \lim_{x\to 0} \frac{3x+\sin x}{x^2+2x} \to \frac{0}{0}$	$L = \lim_{x\to 0} \frac{3+\cos x}{2x+2} = \frac{3+1}{0+2} = 2$	

$0\cdot\pm\infty$ Guidelines	$L = \lim_{x\to a} f(x)\,g(x) \to 0\cdot\pm\infty$	$L = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} \to \frac{0}{0}$	$L = \lim_{x\to a} \frac{D_x f(x)}{D_x\left(\frac{1}{g(x)}\right)}$
Example	$L = \lim_{x\to 0^+} x\ln x \to 0\cdot -\infty$	$L = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \to \frac{-\infty}{\infty}$	$L = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)}$
	$L = \lim_{x\to 0^+} \frac{1}{x} \div \frac{-1}{x^2} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{-x^2}{1} = \lim_{x\to 0^+} (-x) = 0$		

Numerator degree = Denominator degree درجة البسط تساوي درجة المقام	$\lim_{x\to\infty} \frac{4x^3-5x+1}{7x^3+2x-3} = \frac{4}{7}$	معامل اكبر أس في البسط / معامل اكبر أس في المقام
Numerator degree > Denominator degree درجة البسط أكبر من درجة المقام	$\lim_{x\to\infty} \frac{2x^5-3x+11}{27x^2+9x-5} = \infty$	∞
Numerator degree < Denominator degree درجة البسط أصغر من درجة المقام	$\lim_{x\to\infty} \frac{3x^4+7x+11}{7x^5-x-4} = 0$	0

Example 1 Find $\lim_{x\to 0}\frac{\cos 3x - \cos 5x}{x^2}$ 20 November 2006 A

Solution

$$L = \lim_{x\to 0}\frac{\cos 3x - \cos 5x}{x^2} = \frac{1-1}{0} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{-3\sin 3x + 5\sin 5x}{2x} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{-9\cos 3x + 25\cos 5x}{2} = \frac{-9(1) + 25(1)}{2} = \frac{16}{2} = 8$$

Example 2 (2 pts) Find the following limits. $\lim_{x\to 0^+}\frac{x\sinh x}{1-\cosh x}$ 34 July 9, 2011

Solution

$$L = \lim_{x\to 0^+}\frac{x\sinh x}{1-\cosh x} = \frac{0(0)}{1-1} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0^+}\frac{\sinh x + x\cosh x}{-\sinh x} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0^+}\frac{\cosh x + \cosh x + x\sinh x}{-\cosh x} = \frac{1+1+0}{-1} = -2$$

Example 3 Find $\lim_{x\to\infty}(x\sin\frac{1}{x} + e^{-x})$ SECOND SEMESTER 87/88

Solution

$$L = \lim_{x\to\infty} x\sin\frac{1}{x} + \lim_{x\to\infty} e^{-x}$$

$$L_1 = \lim_{x\to\infty} e^{-x} = 0$$

$$L_2 = \lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to\infty}\frac{\frac{-1}{x^2}\cos\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\cos\frac{1}{x} = \cos 0 = 1$$

$$L = L_1 + L_2 = 1 + 0 = 1$$

Example 4

35 2 November 2011

Find the limit if it exists $\lim_{x\to 0}\frac{\ln(x+e^x)}{e^{\sin x}-\cos x}$ (4 pts)

Solution

$$L=\lim_{x\to 0}\frac{\ln(x+e^x)}{e^{\sin x}-\cos x}=\frac{\ln 1}{1-1}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 0}\frac{\frac{1+e^x}{x+e^x}}{e^{\sin x}\cos x+\sin x}=\frac{\frac{1+1}{0+1}}{e^0(1)+0}=\frac{2}{1}=2$$

Example 5

Find $\lim_{x\to\infty} x(\ln(x-1)-\ln(x))$

7 July 1997

Solution

$$L=\lim_{x\to\infty} x|\ln(x-1)-\ln(x)|=\infty(\infty-\infty)$$

$$\lim_{x\to\infty} x(\ln(x-1)-\ln(x))=\lim_{x\to\infty}\frac{\ln\left(\frac{x-1}{x}\right)}{\frac{1}{x}}=\frac{\ln 1}{0}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to\infty}\frac{\frac{x}{x-1}\left[\frac{x-(x-1)}{x^2}\right]}{\frac{-1}{x^2}}=\lim_{x\to\infty}\frac{\frac{1}{x(x-1)}}{\frac{-1}{x^2}}=\lim_{x\to\infty}\frac{-x^2}{x^2-x}=-1$$

Example 6

Find $\lim_{x\to 0}\frac{\sin^{-1}(1-\cos x)}{x^2}$

9 October 1998

Solution

$$L=\lim_{x\to 0}\frac{\sin^{-1}(1-\cos x)}{x^2}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 0}\frac{\frac{\sin x}{\sqrt{1-(1-\cos x)^2}}}{2x}=\lim_{x\to 0}\frac{\sin x}{2x\sqrt{1-1+2\cos x-\cos^2 x}}=\lim_{x\to 0}\frac{\sin x}{2x\sqrt{2\cos x-\cos^2 x}}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 0}\frac{\cos x}{2\sqrt{2\cos x-\cos^2 x}+2x\cdot\frac{-2\sin x+2\cos x\sin x}{2\sqrt{2\cos x-\cos^2 x}}}=\frac{1}{2+0}=\frac{1}{2}$$

حل آخر

$$=\lim_{x\to 0}\frac{\sin x}{2x\sqrt{2\cos x-\cos^2 x}}=\lim_{x\to 0}\frac{1}{2}\cdot\frac{\sin x}{x}\cdot\frac{1}{\sqrt{2\cos x-\cos^2 x}}=\frac{1}{2}\cdot 1\cdot\frac{1}{\sqrt{2-1}}=\frac{1}{2}$$

Example 7

Find $\lim_{x\to 0^+} \frac{\ln(\arcsin x)}{\ln(\sin x)}$ 14 December 1998

Solution

$$L = \lim_{x\to 0^+} \frac{\ln(\sin^{-1} x)}{\ln(\sin x)} = \frac{-\infty}{-\infty}$$

$$\overset{LR}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin^{-1} x}\cdot\frac{1}{\sqrt{1-x^2}}}{\frac{\cos x}{\sin x}} = \lim_{x\to 0^+} \frac{\sin x}{\sin^{-1} x \cos x\sqrt{1-x^2}} = \lim_{x\to 0^+} \frac{\tan x}{\sin^{-1} x\sqrt{1-x^2}} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0^+} \frac{\sec^2 x}{\frac{-2x}{2\sqrt{1-x^2}}\sin^{-1} x + \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}} = \frac{1}{0+1} = 1$$

Example 8

(3 pts) Evaluate the limit $\lim_{x\to\infty} \frac{3x+\ln(\cosh(x))}{x+\ln(x+1)}$ 40 August 7 , 2011

Solution

$$\lim_{x\to\infty} \frac{3x+\ln(\cosh(x))}{x+\ln(x+1)} = \frac{\infty}{\infty}$$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{3+\frac{\sinh x}{\cosh x}}{1+\frac{1}{x+1}} = \lim_{x\to\infty} \frac{3+\tanh x}{1+\frac{1}{x+1}} = \frac{3+1}{1+0} = 4$$

Example 9

Evaluate $\lim_{x\to 0} \frac{x\tan^{-1} x}{\sin^{-1} x}$ 26 January 2006 A

Solution

$$L = \lim_{x\to 0} \frac{x\tan^{-1} x}{\sin^{-1} x} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0} \frac{\tan^{-1} x + \frac{x}{x^2+1}}{\frac{1}{\sqrt{1-x^2}}} = \frac{0+0}{1} = 0$$

Example 10

Evaluate $\lim_{x\to \frac{\pi}{2}^-} \frac{\sin^{-1}(\cos x)}{\ln(\sin x)}$ 33 June 2009 A

Solution

$$L = \lim_{x\to \frac{\pi}{2}^-} \frac{\sin^{-1}(\cos x)}{\ln(\sin x)} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{-\sin x}{\sqrt{1-\cos^2 x}}}{\frac{\cos x}{\sin x}} = \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{-\sin x}{\sin x}}{\frac{\cos x}{\sin x}} = \lim_{x\to \frac{\pi}{2}^-} \frac{-1}{\frac{\cos x}{\sin x}} = \lim_{x\to \frac{\pi}{2}^-} \frac{-\sin x}{\cos x} = \lim_{x\to \frac{\pi}{2}^-} -\tan x = -\infty$$

Homework

1	Find	$\lim_{x\to\infty} e^x \ln(1+e^{-x})$	10 March 1999
2	Find	$\lim_{x\to 0} \frac{x-\tan^{-1} x}{x^2}$	11 October 1999
3	Find the following limits	$\lim_{x\to 1} \frac{\ln(e\,x)-e^{x-1}}{(x-1)^2}$	13 March 2001 A
4	Find	$\lim_{x\to 0} \frac{\ln(\cos x)}{x\,\sin x}$	23 November 2007 A
5	Evaluate b such that	$\lim_{x\to 0} \frac{\sin 3x-3x+bx^3}{x^3}=0$	
6	Evaluate	$\lim_{x\to 0^+} \left(\frac{1}{\tan x}-\frac{1}{x}\right)$	19 May 2001
7	Evaluate	$\lim_{x\to\infty} \tan^{-1}(\ln x-\sinh x)$	21 January 2004 A
8	Find	$\lim_{x\to 0} \frac{\tan^{-1}(x^2)}{1-\cos x}$	10 March 1999
9	Evaluate	$\lim_{x\to 0} \frac{x+\sin x}{\tan^{-1} x}$	May 1999 A
10	Find the following limits	$\lim_{x\to 0^+} \left(\frac{1}{x}-\csc x\right)$	26 July 2008 A
11	Evaluate	$\lim_{x\to \frac{\pi}{2}^+} (\sec x-\tan x)$	33 May 2004
12	Evaluate	$\lim_{x\to 0^+} \left(\frac{3^{\pi x}}{\tan x}-\frac{\cos^2 x}{\sin x}\right)$	35 December 2004 A
13	Find	$\lim_{x\to\infty} \frac{\ln\sqrt{x+10}}{\ln\sqrt{2x+4}}$	5 October 1996
14	Evaluate	$\lim_{x\to 0} \frac{xe^{3x}-x}{1-\cos 2x}$	2 May 1995
15	Evaluate	$\lim_{x\to 0} \left(\frac{1}{x^2}-\frac{1}{x^2\sec x}\right)$	9 May 1997

Homework

16	Evaluate	$\lim_{x\to\infty} \ln \frac{2+e^x}{3x}$	9 May 1997
17	Evaluate	$\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{x}{\ln x}\right)$	13 May 1998
18	Evaluate	$\lim_{x\to 0^+} \frac{\cot x}{\ln x}$	25 December 2001
19	Evaluate	$\lim_{x\to 0^+} (\tan^{-1} x) \csc x$	37 June 2005
20	Evaluate	$\lim_{x\to 1^+} \left(\frac{1}{\ln x} + \frac{1}{1-x}\right)$	39 December 2005
21	Evaluate	$\lim_{x\to\infty} \ln \frac{(\ln x)^2}{\sqrt{x}}$	40 May 2006
22	Evaluate	$\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\tan(x-1)}\right)$	14 November 1998
23	Evaluate	$\lim_{x\to 0^+} (\coth x - \cot x)$	15 December 1998
24	Evaluate the limit , if it exists	$\lim_{x\to 0} \frac{e^{-x^2}-1}{x \sin x}$	30 April 11, 2010
25	Evaluate the following.	$\lim_{x\to 0} \left(\frac{e^x}{\sin x} - \frac{1}{x}\right)$ [2 mark]	31 10 July 2010
26	(2 points) Evaluate the following limit	$\lim_{x\to 0} \frac{\cos(2x) - e^{3x^2}}{x^2}$	50 Dec. 15, 2009
27	(3 pts.) Find the following limit, if it exist	$\lim_{x\to 0} \frac{x - \sin^{-1} x}{1 - \cosh x}$	53 11 December 2010
28	Evaluate the following limit, if it exists	$\lim_{x\to\infty} \tan^{-1}\left(\frac{\sqrt{x^2+1}}{\ln x} + 1\right)$	35 January 24, 2010
29	Evaluate	$\lim_{x\to\infty} \frac{\pi - 2\tan^{-1} x}{\pi - 2\sec^{-1} x}$ (3 points))	37 August 7, 2010
30	(3 pts.) Evaluate the following limit	$\lim_{x\to 3} \frac{(2^x - 3)\tan^{-1}(x-3)}{\log_3(x) - 1}$	38 Jan. 22, 2011

Homework

31	Find $\lim_{x\to 0^+}\left(\coth x-\frac{1}{x}\right)$ if it exist	(4 pts)	14 June 4 , 2011
32	Find $\lim_{x\to e}\frac{\ln(\ln x)}{x-e}$		SECOND SEMESTER 87/88
33	Evaluate $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$		20 January 2001 A
34	Find $\lim_{x\to 0}\frac{\arctan x^2}{\sqrt{x^2+1}-1}$		5 October 1996

32 Find $\lim_{x\to e}\frac{\ln(\ln x)}{x-e}$ SECOND SEMESTER 87/88

Solution

$$L=\lim_{x\to e}\frac{\ln(\ln x)}{x-e}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to e}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}=\frac{1}{1}\cdot\frac{1}{e}=e^{-1}$$

33 Evaluate $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ 20 January 2001 A

Solution

$$L=\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\infty-\infty$$

$$=\lim_{x\to 1^+}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 1^+}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1^+}\frac{1-\frac{1}{x}}{\ln x+1-\frac{1}{x}}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 1^+}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{2}$$

34 Find $\lim_{x\to 0}\frac{\arctan x^2}{\sqrt{x^2+1}-1}$ 5 October 1996

Solution

$$L=\lim_{x\to 0}\frac{\tan^{-1}x^2}{\sqrt{x^2+1}-1}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 0}\frac{\frac{2x}{1+x^4}}{\frac{2x}{2\sqrt{x^2+1}}}=\lim_{x\to 0}\frac{2x\cdot 2\sqrt{x^2+1}}{2x(1+x^4)}=\lim_{x\to 0}\frac{2\sqrt{x^2+1}}{1+x^4}=\frac{2}{1}=2$$